Convergence of Geometric Series

As we add more and more terms to a series, there are several possible outcomes to describe the end-behavior of the sequence.

**Partial Sums Increasing**

With some series, as more terms are added the partial sums just keep getting bigger:

\[1 + 2 + 3 + 4 + 5 + 6 + \ldots \text{ (arithmetic w/ } d = 1)\]

And the partial sums look like:

<table>
<thead>
<tr>
<th>n</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>S_n</td>
<td>1</td>
<td>3</td>
<td>6</td>
<td>10</td>
<td>15</td>
<td>21</td>
<td>28</td>
<td>36</td>
<td>45</td>
</tr>
</tbody>
</table>

Using the language of end-behavior we studied earlier in the year, we’d say that “as \( n \) gets bigger and bigger, \( S_n \) also gets bigger and bigger positive.

**Partial Sums Decreasing**

With some series, as more terms are added the partial sums just keep getting bigger negative:

\[(-2) + (-4) + (-8) + (-16) + (-32) + (-64) + \ldots \text{ (geometric w/ } r = 2)\]

And the partial sums look like:

<table>
<thead>
<tr>
<th>n</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>S_n</td>
<td>-2</td>
<td>-6</td>
<td>-14</td>
<td>-30</td>
<td>-62</td>
<td>-126</td>
<td>-254</td>
<td>-510</td>
<td>-1024</td>
</tr>
</tbody>
</table>

Using the language of end-behavior we studied earlier in the year, we’d say that “as \( n \) gets bigger and bigger, \( S_n \) also gets bigger and bigger negative.

**Partial Sums Oscillating**

With some series, as more terms are added the partial sums just flip between two values:

\[(-1) + (1) + (-1) + (1) + (-1) + (1) + \ldots \text{ (geometric w/ } r = -1)\]

And the partial sums look like:

<table>
<thead>
<tr>
<th>n</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>S_n</td>
<td>-1</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>-1</td>
</tr>
</tbody>
</table>

Using the language of end-behavior we studied earlier in the year, we’d say that “as \( n \) gets bigger and bigger, \( S_n \) flips between \(-1\) and \(0\).
Convergence of Geometric Series (12.5)

Partial Sums Converge

With some series, as more terms are added the partial sums approach a single value.

\[ (4) + (2) + (1) + (1/2) + (1/4) + (1/8) + \ldots \] (geometric w/ \( r = 1/2 \))

And the partial sums look like:

<table>
<thead>
<tr>
<th>( n )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S_n )</td>
<td>4</td>
<td>6</td>
<td>7</td>
<td>7.5</td>
<td>7.75</td>
<td>7.875</td>
<td>7.9375</td>
<td>7.96875</td>
<td>7.984375</td>
</tr>
</tbody>
</table>

From this table of values, it is unclear whether the partial sums will keep getting larger and larger forever (albeit very slowly) or whether they are approaching a single number which they will never surpass. Since this is a geometric series, let’s find \( S_n \) for some really large values of \( n \) using our formula from the previous section.

\[ S_n = \frac{t_1(1-r^n)}{1-r}, \quad \text{if} \ n = 10, \quad S_{10} = \frac{4(1-(1/2)^{10})}{1-(1/2)} = 7.9921875 \]

\[ \text{if} \ n = 25, \quad S_{25} = \frac{4(1-(1/2)^{25})}{1-(1/2)} \approx 7.999999762 \]

It certainly looks like the partial sums are approaching 8. Looking at a graph of the function \( S_n = \frac{4(1-(1/2)^n)}{1-(1/2)} \), we see that it has a horizontal asymptote at 8.

Using the language of end-behavior we studied earlier in the year, we’d say that “as \( n \) gets bigger and bigger, \( S_n \) gets closer and closer to 8.” This situation is called convergence.

**Definition** – Convergence: A series converges if as \( n \) gets bigger and bigger, the partial sum \( S_n \) approaches a single finite value.

The question then arises: Well, which series converge?

To answer this, think about the examples above. There is one property that must be met to even have a chance, namely: each term must be smaller than the previous term.
Though this is not the only condition (you’ll study this topic in greater depth in future courses), only one type of series that we’ve studied has this property – geometric series with common ratio \( r \), with \( r \) being a fraction less than 1.

In fact, we can see that geometric series (with \(-1 < r < 1\)) converge based on what happens to the partial sum formula:

\[
S_n = \frac{t_1(1-r^n)}{1-r},
\]

this represent the sum of the first \( n \) terms in the series.

- as \( n \) gets bigger and bigger, \( r^n \) gets smaller and smaller\(^1\), approaching 0.
- therefore as \( n \) gets bigger and bigger, \( 1 - r^n \) approaches 1.

So, as \( n \) gets bigger and bigger, \( S = \frac{t_1}{1-r} \). Notice that we dropped the \( S_n \) notation in favor of \( S \) for this final formula – this should be interpreted as “the sum of all the terms” or the “total sum of the infinite series.”

**Example 1**: Returning to the series from above, find the sum of all the terms in the series

\[
(4) + (2) + (1) + (1/2) + (1/4) + (1/8) + (1/16) + \ldots
\]

Since this is a geometric series with \( r = 1/2 \), the sum of all the terms in the series can be found with the formula

\[
S = \frac{t_1}{1-r} = \frac{4}{1-(1/2)} = \frac{4}{1/2} = 8,
\]

which is exactly the value we suspected for this series all along!

**Exercises**

1. Determine whether or not the series converges. If it converges, find the sum of all its terms. If it does not converge, write “Does not converge!”
   
   a. \( 2 + 1 + 0 + (-1) + (-2) + (-3) + \ldots \)
   
   b. \( 1 + 1/3 + 1/9 + 1/27 + \ldots \)

   c. \( 10 + 5 + 5/2 + 5/4 + 5/8 + \ldots \)

\(^1\) Don’t take our word for it. Convince yourself of this fact by doing Exercise 2.
Convergence of Geometric Series (12.5)

d. \[ \sum_{n=1}^{\infty} 4(\frac{2}{5})^{n-1} \]

(Hint: This is just the sigma-notation for “sum all the terms.” You should write the first several terms of the series until you recognize its type)

e. \[ \sum_{n=1}^{\infty} 4(\frac{5}{2})^{n-1} \]

f. \[ \sum_{n=1}^{\infty} 4 + 2^n \]

2. Fill in the table of values for each of the tables below

<table>
<thead>
<tr>
<th>n</th>
<th>((1/2)^n)</th>
<th>n</th>
<th>((-1/2)^n)</th>
<th>n</th>
<th>((9/10)^n)</th>
<th>n</th>
<th>((2)^n)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1/2</td>
<td>1</td>
<td>-1/2</td>
<td>1</td>
<td>0.9</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>1/4</td>
<td>2</td>
<td>1/4</td>
<td>2</td>
<td>0.81</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>5</td>
<td></td>
<td>5</td>
<td>5</td>
<td>5</td>
<td></td>
<td>5</td>
<td></td>
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<tr>
<td>10</td>
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<tr>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td></td>
<td>100</td>
<td></td>
</tr>
</tbody>
</table>

Based on your result in the table above, consider the following statements and answer the questions

“As \( n \) gets bigger and bigger, \( r^n \) gets smaller and smaller, approaching 0.”

Considering all real numbers, for what possible values of \( r \) is this statement true?
3. Consider the decimal: $0.\overline{3}$.

Every repeating decimal can be written as a series. $0.\overline{3}$ can be written as:

\[ 0.3 + 0.03 + 0.003 + 0.0003 + \ldots \]

a. What type of series is this?

b. What is the sum of all the terms in this series? Surprised?

c. Use the method outlined here to find the fraction equivalent of the repeating decimal $0.\overline{12}$. 